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APPROXIMATE SOLUTION OF PLANE PROBLEMS FOR IDEAL ELASTOPLASTIC INHOMOGENEOUS BODIES

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The theory of plasticity of inhomogeneous bodies was described by Olszak et al. and by Grigor'ev in [1, 2] in which the related bibliography is given. In the present paper, the algorithm of successive approximations for determination of a stress-strain state of bodies from an ideal elastoplastic material, which was elaborated by Ivlev and Ershov [3], is extended to the case of inhomogeneous bodies.

The Mises yield condition for inhomogeneous bodies is formulated as the limiting value of the energy of elastic straining. As an example, we consider the biaxial tension of a thick plate weakened by a round hole.

1. For the plane problem, we write the elastic potential for an elastic isotropic incompressible body in the form [4]

$$W = \frac{1}{8G}[(\sigma_\rho - \sigma_\theta)^2 + 4\tau_{\rho\theta}^2], \tag{1.1}$$

where W is the elastic potential, G is the modulus of shear; σ_ρ, σ_θ, and $\tau_{\rho\theta}$ are the stress components in the polar coordinate system $\rho\theta$. It is evident that the quantity W determines the strain energy of the material.

The material is assumed to be inhomogeneous:

$$G = G(\rho, \theta). \tag{1.2}$$

According to (1.1), we obtain

$$e_\rho^e = \frac{1}{4G}(\sigma_\rho - \sigma_\theta), \qquad e_\theta^e = -e_\rho^e, \qquad e_{\rho\theta}^e = \frac{1}{2G}\tau_{\rho\theta}. \tag{1.3}$$

Here e_ρ^e, e_θ^e, and $e_{\rho\theta}^e$ are the elastic strains subject to the compatibility condition

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\Big(\rho^2\frac{\partial e_\theta}{\partial\rho}\Big) + \frac{1}{\rho}\frac{\partial^2 e_\rho}{\partial\theta^2} - \frac{\partial e_\rho}{\partial\rho} = \frac{2}{\rho}\frac{\partial^2}{\partial\rho\partial\theta}(\rho e_{\rho\theta}). \tag{1.4}$$

We assume [5] that the plastic state occurs when the strain energy reaches some constant value:

$$W = \frac{1}{8G}[(\sigma_\rho - \sigma_\theta)^2 + 4\tau_{\rho\theta}^2] = k^2, \qquad k = \text{const}. \tag{1.5}$$

In accordance with (1.2), we rewrite relation (1.5) in the form

$$(\sigma_\rho - \sigma_\theta)^2 + 4\tau_{\rho\theta}^2 - 8k^2G(\rho, \theta) = 0. \tag{1.6}$$

By relation (1.6), the yield point depends on the character of inhomogeneity when $\sigma_\rho = 2k\sqrt{2G(\rho)}$ and $\sigma_\theta = \tau_{\rho\theta} = 0$.

Regarding relation (1.6) as a plastic potential, we obtain

$$\varepsilon_\rho^p = \lambda\frac{\partial f}{\partial\sigma_\rho}, \qquad \varepsilon_\theta^p = \lambda\frac{\partial f}{\partial\sigma_\theta}, \qquad \varepsilon_{\rho\theta}^p = \frac{\lambda}{2}\frac{\partial f}{\partial\tau_{\rho\theta}}, \qquad \lambda \geqslant 0,$$

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where ε_ρ^p, ε_θ^p, and $\varepsilon_{\rho\theta}^p$ are the components of the plastic strain velocity. We write the modulus expression (1.2) as

$$G(\rho,\theta)=\sum_{n=0}^{\infty}\delta^n G_n(\rho,\theta),\quad G_0=\text{const}, \tag{1.7}$$

where δ is a small parameter.

The solution is constructed as the series

$$\sigma_{ij}=\sum_{n=0}^{\infty}\delta^n\sigma_{ij}^{(n)},\quad e_{ij}=\sum_{n=0}^{\infty}\delta^n e_{ij}^{(n)},\quad \ldots\ . \tag{1.8}$$

Let us determine the first approximation. From (1.6)–(1.8), we have

$$\sigma_\rho^0-\sigma_\theta^0=2\eta k\sqrt{2G_0},\quad \sigma_\rho'-\sigma_\theta'=\eta k G_1\sqrt{\frac{2}{G_0}},\quad \eta=\operatorname{sign}(\sigma_\rho^0-\sigma_\theta^0), \tag{1.9}$$

where the superscript 0 refers to the "unperturbed"-state components and the prime refers to the "perturbed"-state components.

The equilibrium equations

$$\frac{\partial\sigma_\rho'}{\partial\rho}+\frac{1}{\rho}\frac{\partial\tau_{\rho\theta}'}{\partial\theta}+\frac{\sigma_\rho'-\sigma_\theta'}{\rho}=0,\qquad \frac{\partial\tau_{\rho\theta}'}{\partial\rho}+\frac{1}{\rho}\frac{\partial\sigma_\theta'}{\partial\theta}+\frac{2\tau_{\rho\theta}'}{\rho}=0 \tag{1.10}$$

are satisfied assuming that [3]

$$\sigma_\rho'=\frac{1}{\rho}\frac{\partial\Phi'}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\Phi'}{\partial\theta^2},\quad \sigma_\theta'=\frac{\partial^2\Phi'}{\partial\rho^2},\quad \tau_{\rho\theta}'=-\frac{\partial}{\partial\rho}\Big(\frac{1}{\rho}\frac{\partial\Phi'}{\partial\theta}\Big). \tag{1.11}$$

We find from (1.9) and (1.11) that

$$\frac{\partial^2\Phi'}{\partial\rho^2}-\frac{1}{\rho^2}\frac{\partial^2\Phi'}{\partial\theta^2}-\frac{1}{\rho}\frac{\partial\Phi'}{\partial\rho}=-\eta k\sqrt{\frac{2}{G_0}}G_1. \tag{1.12}$$

The general solution of Eq. (1.12) can be written as the sum of the solution of the homogeneous equation Φ_0' and the particular solution of the nonhomogeneous equation Φ_1'. According to [3],

$$\Phi_0'=H_{01}+\rho^2H_{02}+\rho\cos\theta(H_{11}+H_{12}\ln\rho) \tag{1.13}$$

$$+\rho\sum_{m=2}^{\infty}[H_{m1}\cos(\sqrt{m^2-1}\ln\rho)+H_{m2}\sin(\sqrt{m^2-1}\ln\rho)]\cos m\theta.$$

We present G_1 as the expansion

$$G_1=\sum_{m=0}^{\infty}f_m(\rho)\cos m\theta. \tag{1.14}$$

We search for a particular solution of Eq. (1.12) with the right-hand side (1.14) in the form

$$\Phi_1'=\sum_{m=0}^{\infty}F_m(\rho)\cos m\theta. \tag{1.15}$$

Substituting expansions (1.14) and (1.15) into Eq. (1.12), we have

$$F_m''(\rho)-\frac{1}{\rho}F_m'(\rho)+\frac{m^2}{\rho^2}F_m(\rho)=-\eta k\sqrt{\frac{2}{G_0}}f_m(\rho). \tag{1.16}$$

It follows from (1.16) that

$$F_m(\rho)=\frac{\eta k}{\sqrt{2G_0}}\Big[\int\rho f_0(\rho)\,d\rho-\rho^2\int\frac{f_0(\rho)}{\rho}d\rho\Big]+\eta k\rho\sqrt{\frac{2}{G_0}}\Big[\int\ln\rho f_1(\rho)\,d\rho-\ln\rho\int f_1(\rho)\,d\rho\Big]$$

$$+\eta k\rho\sqrt{\frac{2}{G_0}}\sum_{m=2}^{\infty}\frac{1}{\sqrt{m^2-1}}\Big[\cos(\sqrt{m^2-1}\ln\rho)\int f_m(\rho)\sin(\sqrt{m^2-1}\ln\rho)\,d\rho$$
$$-\sin(\sqrt{m^2-1}\ln\rho)\int f_m(\rho)\cos(\sqrt{m^2-1}\ln\rho)\,d\rho\Big].$$

Using the general solution of Eq. (1.12), from (1.11) we obtain the relations for the stress components in the plastic region:

$$\sigma_\rho'^p = 2H_{02}-\eta k\sqrt{\frac{2}{G_0}}\int\frac{f_0(\rho)}{\rho}\,d\rho+\frac{\cos\theta}{\rho}\Big[H_{12}-\eta k\sqrt{\frac{2}{G_0}}\int f_1(\rho)\,d\rho\Big]$$
$$+\frac{1}{\rho}\sum_{m=2}^{\infty}[\cos(\sqrt{m^2-1}\ln\rho)((1-m^2)H_{m1}+\sqrt{m^2-1}H_{m2})$$
$$+\sin(\sqrt{m^2-1}\ln\rho)(-\sqrt{m^2-1}H_{m1}+(1-m^2)H_{m2})]\cos m\theta$$
$$-\eta k\sqrt{\frac{2}{G_0}}[\{\sqrt{m^2-1}\cos(\sqrt{m^2-1}\ln\rho)+\sin(\sqrt{m^2-1}\ln\rho)\}\int f_m(\rho)\sin(\sqrt{m^2-1}\ln\rho)d\rho$$
$$-\{\sqrt{m^2-1}\sin(\sqrt{m^2-1}\ln\rho)-\cos(\sqrt{m^2-1}\ln\rho)\}\int f_m(\rho)\cos(\sqrt{m^2-1}\ln\rho)\,d\rho]\cos m\theta,$$
$$\sigma_\theta'^p = 2H_{02}-\eta k\sqrt{\frac{2}{G_0}}\Big[\int\frac{f_0(\rho)}{\rho}\,d\rho+f_0(\rho)\Big]+\frac{\cos\theta}{\rho}\Big[H_{12}-\eta k\sqrt{\frac{2}{G_0}}\Big(\int f_1(\rho)\,d\rho-\rho f_1(\rho)\Big)\Big]$$
$$+\frac{1}{\rho}\sum_{m=2}^{\infty}[\cos(\sqrt{m^2-1}\ln\rho)((1-m^2)H_{m1}+\sqrt{m^2-1}H_{m2})+\sin(\sqrt{m^2-1}\ln\rho)(-\sqrt{m^2-1}H_{m1}$$
$$+(1-m^2)H_{m2})]\cos m\theta-\eta k\sqrt{\frac{2}{G_0}}[\{\sqrt{m^2-1}\cos(\sqrt{m^2-1}\ln\rho)\tag{1.17}$$
$$+\sin(\sqrt{m^2-1}\ln\rho)\}\int f_m(\rho)\sin(\sqrt{m^2-1}\ln\rho)\,d\rho-\{\sqrt{m^2-1}\sin(\sqrt{m^2-1}\ln\rho)$$
$$-\cos(\sqrt{m^2-1}\ln\rho)\}\int f_m(\rho)\cos(\sqrt{m^2-1}\ln\rho)\,d\rho-\rho f_m(\rho)]\cos m\theta,$$
$$\tau_{\rho\theta}'^p=\frac{\sin\theta}{\rho}\Big[H_{12}-\eta k\sqrt{\frac{2}{G_0}}\int f_1(\rho)\,d\rho\Big]+\frac{1}{\rho}\sum_{m=2}^{\infty}[m\sqrt{m^2-1}(-H_{m1}\sin(\sqrt{m^2-1}\ln\rho)$$
$$+H_{m2}\cos(\sqrt{m^2-1}\ln\rho))]\sin m\theta-\eta km\sqrt{\frac{2}{G_0}}\Big[\sin(\sqrt{m^2-1}\ln\rho)\int f_m(\rho)\sin(\sqrt{m^2-1}\ln\rho)d\rho$$
$$+\cos(\sqrt{m^2-1}\ln\rho)\int f_m(\rho)\cos(\sqrt{m^2-1}\ln\rho)d\rho\Big]\sin m\theta.$$

The stresses in the elastic region are found from the nonhomogeneous biharmonic equation

$$\nabla\nabla\Phi'=\eta\Psi(\rho,\theta),\tag{1.18}$$

where the right-hand side is determined, according to (1.3), (1.4), (1.7), and (1.8), as follows:

$$\Psi=\frac{\rho^4(\sigma_\theta^0-\sigma_\rho^0)}{G_0}\frac{\partial^2G_1}{\partial\rho^2}-\frac{\rho^2(\sigma_\theta^0-\sigma_\rho^0)}{G_0}\frac{\partial^2G_1}{\partial\theta^2}+\frac{\rho^4G_1}{G_0}\frac{\partial^2}{\partial\rho^2}(\sigma_\theta^0-\sigma_\rho^0)$$
$$+\Big[\frac{2\rho^4}{G_0}\frac{\partial G_1}{\partial\rho}+\frac{3\rho^3G_1}{G_0}\Big]\frac{\partial}{\partial\rho}(\sigma_\theta^0-\sigma_\rho^0)+\frac{3\rho^3(\sigma_\theta^0-\sigma_\rho^0)}{G_0}\frac{\partial G_1}{\partial\rho}.\tag{1.19}$$

Let us represent Ψ as the expansion

$$\Psi = \sum_{m=0}^{\infty} \Psi_m(\rho) \cos m\theta. \tag{1.20}$$

The solution of the homogeneous equation (1.18) is known [4]. The particular solution of Eq. (1.18) is found similarly to (1.15). Using the general solution (1.18) and the substitution (1.11), we obtain the relations for the stress components in the elastic region:

$$\begin{aligned}
\sigma_\rho'^e &= 2C_{01} + (2\ln\rho + 1)C_{02} + \frac{1}{\rho^2}C_{04} + \frac{\eta}{4}\Big[-\int \rho\Psi_0(\rho)\,d\rho - 2\int \rho\ln\rho\Psi_0(\rho)\,d\rho \\
&+2\ln\rho\int \rho\Psi_0(\rho)\,d\rho + \frac{1}{\rho^2}\int \rho^3\Psi_0(\rho)\,d\rho\Big] + \Big\{2\rho C_{11} - \frac{2}{\rho^3}C_{12} + \frac{1}{\rho}C_{14}\Big\}\cos\theta \\
&+\frac{\eta}{4}\Big[\frac{\rho}{2}\int \Psi_1(\rho)\,d\rho - \frac{1}{\rho}\int \rho^2\Psi_1(\rho)\,d\rho + \frac{1}{2\rho^3}\int \rho^4\Psi_1(\rho)\,d\rho\Big]\cos\theta \\
&+\sum_{m=2}^{\infty}\{-m(m-1)\rho^{m-2}C_{m1} - m(m+1)\rho^{-m-2}C_{m2} \\
&-(m+1)(m-2)\rho^m C_{m3} - (m+2)(m-1)\rho^{-m}C_{m4}\}\cos m\theta \\
&+\frac{\eta}{8m}\Big[m\rho^{m-2}\int \rho^{-m+1}\Psi_m(\rho)\,d\rho + m\rho^{-m-2}\int \rho^{m+1}\Psi_m(\rho)\,d\rho \\
&-(m-2)\rho^m\int \rho^{-m-1}\Psi_m(\rho)\,d\rho - (m+2)\rho^{-m}\int \rho^{m-1}\Psi_m(\rho)\,d\rho\Big]\cos m\theta, \\
\sigma_\theta'^e &= 2C_{01} + (2\ln\rho + 3)C_{02} - \frac{1}{\rho^2}C_{04} + \frac{\eta}{4}\Big[\int \rho\Psi_0(\rho)\,d\rho - 2\int \rho\ln\rho\Psi_0(\rho)\,d\rho \\
&+2\ln\rho\int \rho\Psi_0(\rho)\,d\rho - \frac{1}{\rho^2}\int \rho^3\Psi_0(\rho)\,d\rho\Big] + \Big\{6\rho C_{11} + \frac{2}{\rho^3}C_{12} + \frac{1}{\rho}C_{14}\Big\}\cos\theta \\
&+\frac{\eta}{4}\Big[\frac{3\rho}{2}\int \Psi_1(\rho)\,d\rho - \frac{1}{\rho}\int \rho^2\Psi_1(\rho)\,d\rho - \frac{1}{2\rho^3}\int \rho^4\Psi_1(\rho)\,d\rho\Big]\cos\theta \\
&+\sum_{m=2}^{\infty}\{m(m-1)\rho^{m-2}C_{m1} + m(m+1)\rho^{-m-2}C_{m2} \\
&+(m+1)(m+2)\rho^m C_{m3} + (-m+1)(-m+2)\rho^{-m}C_{m4}\}\cos m\theta \\
&+\frac{\eta}{8m}\Big[-m\rho^{m-2}\int \rho^{-m+1}\Psi_m(\rho)\,d\rho - m\rho^{-m-2}\int \rho^{m+1}\Psi_m(\rho)\,d\rho \\
&+(m+2)\rho^m\int \rho^{-m-1}\Psi_m(\rho)\,d\rho - (-m+2)\rho^{-m}\int \rho^{m-1}\Psi_m(\rho)\,d\rho\Big]\cos m\theta, \\
\tau_{\rho\theta}'^e &= \Big\{2\rho C_{11} - \frac{2}{\rho^3}C_{12} + \frac{1}{\rho}C_{14}\Big\}\sin\theta + \frac{\eta}{4}\Big[\frac{\rho}{2}\int \Psi_1(\rho)\,d\rho - \frac{1}{\rho}\int \rho^2\Psi_1(\rho)\,d\rho \\
&+\frac{1}{2\rho^3}\int \rho^4\Psi_1(\rho)\,d\rho\Big]\sin\theta + \sum_{m=2}^{\infty}\{m(m-1)\rho^{m-2}C_{m1} - m(m+1)\rho^{-m-2}C_{m2} \\
&+m(m+1)\rho^m C_{m3} - m(m-1)\rho^{-m}C_{m4}\}\sin m\theta + \frac{\eta}{8}\Big[-\rho^{m-2}\int \rho^{-m+1}\Psi_m(\rho)\,d\rho \\
&+\rho^{-m-2}\int \rho^{m+1}\Psi_m(\rho)\,d\rho + \rho^m\int \rho^{-m-1}\Psi_m(\rho)\,d\rho - \rho^{-m}\int \rho^{m-1}\Psi_m(\rho)\,d\rho\Big]\sin m\theta.
\end{aligned} \tag{1.21}$$

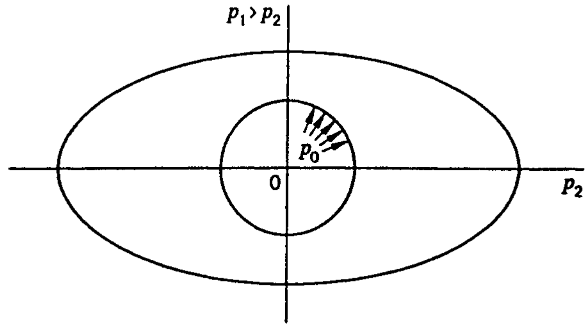

Fig. 1

The radius of plasticity is determined from the conjugation condition [3]

$$\left[\sigma'_\theta + \frac{d\sigma^0_\theta}{d\rho}\rho'_s\right] = 0 \qquad \text{for} \qquad \rho = 1. \tag{1.22}$$

2. Let us consider an infinite plane with a round hole of radius a which is stretched at infinity by two mutually perpendicular tensile stresses p_1 and p_2, the normal pressure p_0 (see Fig. 1) acting on the hole contour. For an ideal isotropic elastoplastic body, this problem was considered by Galin [6].

We determine the solution near a certain "unperturbed" state by the small-parameter method. As the parameter, we use the quantity $\delta = (p_1 - p_2)/2k$.

Below, all quantities that have the dimension of stress will be referred to the yield point k, and the quantities that have the dimension of length will be referred to the radius of the plastic "unperturbed"-state zone r_{0s}. We denote $\sigma_\rho = \sigma_r/k$, $\tau_{\rho\theta} = \tau_{r\theta}/k$, $\rho = r/r_{0s}$, $\alpha = a/r_{0s}$, and $q_i = p_i/k$, where $i = 0$, 1, and 2. For the nondimensional quantities σ_θ/k and G/k, we use the former designations σ_θ and G.

For $0 \leqslant \delta < 1$, the plate is in an elastoplastic state. We need to determine the stress-strain state of the plate and the boundary between the elastic and plastic zones.

According to [3], the linearized boundary conditions are of the form

$$\sigma_\rho^{\infty e} = q - \delta\cos 2\theta, \quad \sigma_\theta^{\infty e} = q + \delta\cos 2\theta, \quad \tau_{\rho\theta}^{\infty e} = \delta\sin 2\theta, \quad q = (q_1 + q_2)/2 \tag{2.1}$$

at infinity and

$$\sigma_\rho^p = -q_0, \qquad \tau_{\rho\theta}^p = 0 \qquad \text{for} \qquad \rho = \alpha \tag{2.2}$$

on the hole contour.

Let us consider a zero approximation, i.e., the limiting case where $\delta = 0$. As the "unperturbed" state, we choose the axisymmetric state of the plate with a hole with radius a whose contour is under the normal pressure p_0 and under the uniform pressure $p = p_1 = p_2$ at infinity. The solution of this elastoplastic problem is known [3] and is as follows:

$$\sigma_\rho^{0p} = -q_0 + 2\eta\ln(\rho/\alpha), \quad \sigma_\theta^{0p} = -q_0 + 2\eta[1 + \ln(\rho/\alpha)], \quad \tau_{\rho\theta}^{0p} = 0; \tag{2.3}$$

$$\sigma_\rho^{0e} = q - \eta/\rho^2, \qquad \sigma_\theta^{0e} = q + \eta/\rho^2, \qquad \tau_{\rho\theta}^{0e} = 0. \tag{2.4}$$

In this case, the boundary between the elastic and plastic zones will be a circumference with radius $r_{0s} = a\exp[(|q_0 - q| - 1)/2]$. In the general form, the plastic and elastic solutions are determined by formulas (1.17) and (1.21).

On the hole contour, we have

$$\sigma_\rho'^p = \tau_{\rho\theta}'^p = 0 \qquad \text{for} \qquad \rho = \alpha. \tag{2.5}$$

According to [3], the conjugation conditions take the form

$$\sigma_\rho'^p = \sigma_\rho'^e, \qquad \tau_{\rho\theta}'^p = \tau_{\rho\theta}'^e \qquad \text{for} \qquad \rho = 1. \tag{2.6}$$

According to (2.1), at infinity

$$\sigma_\rho^{\prime e} = -\delta\cos 2\theta, \qquad \tau_{\rho\theta}^{\prime e} = \delta\sin 2\theta \qquad \text{for} \qquad \rho = \infty. \tag{2.7}$$

Satisfying conditions (2.5)-(2.7), from (1.17) and (1.21) we obtain the relations for the stress components in the plastic zone:

$$\sigma_\rho^{\prime p} = -\sqrt{\frac{2}{G_0}}\int_\alpha^\rho \frac{f_0(\rho)}{\rho}\,d\rho - \frac{1}{\rho}\sqrt{\frac{2}{G_0}}\Big[(\sqrt{3}\cos(\sqrt{3}\ln\rho) + \sin(\sqrt{3}\ln\rho))\int_\alpha^\rho f_2(\rho)\sin(\sqrt{3}\ln\rho)\,d\rho$$

$$-(\sqrt{3}\sin(\sqrt{3}\ln\rho) - \cos(\sqrt{3}\ln\rho))\int_\alpha^\rho f_2(\rho)\cos(\sqrt{3}\ln\rho)\,d\rho\Big]\cos 2\theta,$$

$$\sigma_\theta^{\prime p} = -\sqrt{\frac{2}{G_0}}\Big[\int_\alpha^\rho \frac{f_0(\rho)}{\rho}d\rho + f_0(\rho)\Big] - \frac{1}{\rho}\sqrt{\frac{2}{G_0}}\Big[(\sqrt{3}\cos(\sqrt{3}\ln\rho) + \sin(\sqrt{3}\ln\rho))\int_\alpha^\rho f_2(\rho)\sin(\sqrt{3}\ln\rho)\,d\rho$$

$$-(\sqrt{3}\sin(\sqrt{3}\ln\rho) - \cos(\sqrt{3}\ln\rho))\int_\alpha^\rho f_2(\rho)\cos(\sqrt{3}\ln\rho)\,d\rho + \rho f_2(\rho)\Big]\cos 2\theta, \tag{2.8}$$

$$\tau_{\rho\theta}^{\prime p} = -\frac{2}{\rho}\sqrt{\frac{2}{G_0}}\Big[\sin(\sqrt{3}\ln\rho)\int_\alpha^\rho f_2(\rho)\sin(\sqrt{3}\ln\rho)d\rho + \cos(\sqrt{3}\ln\rho)\int_\alpha^\rho f_2(\rho)\cos(\sqrt{3}\ln\rho)\,d\rho\Big]\sin 2\theta.$$

The relations for the stress components in the elastic zone are of the form

$$\sigma_\rho^{\prime e} = \frac{C_{04}}{\rho^2} + \frac{1}{4}\Big[-\int_\infty^\rho (\rho + 2\rho\ln\rho)\Psi_0(\rho)\,d\rho + 2\ln\rho\int_\infty^\rho \rho\Psi_0(\rho)\,d\rho + \frac{1}{\rho^2}\int_\infty^\rho \rho^3\Psi_0(\rho)\,d\rho\Big]$$

$$+\Big(-1 - \frac{6}{\rho^4}C_{22} - \frac{4}{\rho^2}C_{24} + \frac{1}{8}\Big[\int_\infty^\rho \frac{\Psi_2(\rho)}{\rho}\,d\rho + \frac{1}{\rho^4}\int_\infty^\rho \rho^3\Psi_2(\rho)\,d\rho - \frac{2}{\rho^2}\int_\infty^\rho \rho\Psi_2(\rho)\,d\rho\Big]\Big)\cos 2\theta,$$

$$\sigma_\theta^{\prime e} = -\frac{C_{04}}{\rho^2} + \frac{1}{4}\Big[\int_\infty^\rho (\rho - 2\rho\ln\rho)\Psi_0(\rho)\,d\rho + 2\ln\rho\int_\infty^\rho \rho\Psi_0(\rho)d\rho - \frac{1}{\rho^2}\int_\infty^\rho \rho^3\Psi_0(\rho)\,d\rho\Big]$$

$$+\Big(1 + \frac{6}{\rho^4}C_{22} + \frac{1}{8}\Big[-\int_\infty^\rho \frac{\Psi_2(\rho)}{\rho}\,d\rho - \frac{1}{\rho^4}\int_\infty^\rho \rho^3\Psi_2(\rho)\,d\rho + 2\rho^2\int_\infty^\rho \frac{\Psi_2(\rho)}{\rho^3}\,d\rho\Big]\Big)\cos 2\theta,$$

$$\tau_{\rho\theta}^{\prime e} = \Big(1 - \frac{6}{\rho^4}C_{22} - \frac{2}{\rho^2}C_{24} + \frac{1}{8}\Big[-\int_\infty^\rho \frac{\Psi_2(\rho)}{\rho}d\rho + \frac{1}{\rho^4}\int_\infty^\rho \rho^3\Psi_2(\rho)d\rho + \rho^2\int_\infty^\rho \frac{\Psi_2(\rho)}{\rho^3}\,d\rho - \frac{1}{\rho^2}\int_\infty^\rho !\rho\Psi_2(\rho)\,d\rho\Big]\Big)\sin 2\theta, \tag{2.9}$$

$$C_{04} = -\sqrt{\frac{2}{G_0}}\int_\alpha^1 \frac{f_0(\rho)}{\rho}\,d\rho + \frac{1}{4}\int_1^\infty (\rho^3 - \rho -- 2\rho\ln\rho)\Psi_0(\rho)\,d\rho,$$

$$C_{22} = \frac{1}{2} + \frac{1}{48}\int_1^\infty \Big(\frac{3}{\rho} - \rho^3 - \frac{2}{\rho^3}\Big)\Psi_2(\rho)\,d\rho - \frac{1}{\sqrt{6G_0}}\int_\alpha^1 f_2(\rho)\sin(\sqrt{3}\ln\rho)\,d\rho + \frac{1}{\sqrt{2G_0}}\int_\alpha^1 f_2(\rho)\cos(\sqrt{3}\ln\rho)\,d\rho,$$

$$C_{24} = -1 + \frac{1}{16}\int_1^\infty \Big(\rho + \frac{1}{\rho^3} - \frac{2}{\rho}\Big)\Psi_2(\rho)\,d\rho + \sqrt{\frac{3}{2G_0}}\int_\alpha^1 f_2(\rho)\sin(\sqrt{3}\ln\rho)\,d\rho - \frac{1}{\sqrt{2G_0}}\int_\alpha^1 f_2(\rho)\cos(\sqrt{3}\ln\rho)\,d\rho.$$

From (2.3), (2.4), (2.8), (2.9), and the conjugation condition (1.22), we find the radius of the plastic zone:

$$\rho_s' = \frac{1}{\sqrt{2G_0}} \int_{\alpha}^{1} \frac{f_0(\rho)}{\rho}\, d\rho + \frac{1}{4} \int_{1}^{\infty} \rho \ln \rho \Psi_0(\rho)\, d\rho + \frac{1}{4} f_0(1) + \Big[1 + \frac{1}{32} \int_{1}^{\infty} \Big(\frac{4}{\rho} - \frac{3}{\rho^3}\Big) \Psi_2(\rho)\, d\rho$$

$$+ \frac{1}{\sqrt{2G_0}} \int_{\alpha}^{1} f_2(\rho) \cos(\sqrt{3} \ln \rho)\, d\rho + \frac{1}{4} f_2(1)\Big] \cos 2\theta. \tag{2.10}$$

The particular function of the inhomogeneity is chosen as follows:

$$G_1 = \frac{A}{\rho^5} + \frac{B}{\rho^5} \cos 2\theta, \quad A,\, B = \mathrm{const}, \tag{2.11}$$

where G_1, A, and B are referred to the yield point k. The character of behavior of the inhomogeneity (2.11) is determined from the requirement of convergence of the integrals (2.8)–(2.10). The inhomogeneity can be arbitrary to the finite limit ρ_0. If the inhomogeneity is specified in the form (2.11), the relation for the radius of the plastic zone takes the following form, according to (1.14), (1.19), (1.20), (2.10), and (2.11):

$$\rho_s' = \frac{A}{4} - \frac{A}{10}\sqrt{\frac{2}{G_0}} + \frac{35A}{18G_0} + \frac{A}{10\alpha^5} + \Big[1 + \frac{B}{4} - \frac{2B}{19}\sqrt{\frac{2}{G_0}} + \frac{507B}{560G_0}$$

$$+ \frac{B}{38\alpha^4}\sqrt{\frac{2}{G_0}}(4\cos(\sqrt{3}\ln\alpha) - \sqrt{3}\sin(\sqrt{3}\ln\alpha))\Big] \cos 2\theta. \tag{2.12}$$

It follows from (2.12) that the effect of the inhomogeneity, which is characterized by the quantities A and B in (2.11), depends on the modulus of shear G_0 and on the parameter with radius α. Note that the smaller the G_0 and α, the greater the ρ_s'. For $A = B = 0$, the results of [3] hold true.

Note that material inhomogeneity leads to a change in the stress state in both the plastic (2.8) and elastic (2.9) zones. The displacements can be found according to [3, 7], and the spatial state can be determined according to [8].

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